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ON A CLASS OF GAMES

Samuel Karlin

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# ON A CLASS OF GAMES

Samuel Karlin

*This paper*

The purpose of this note is to describe qualitatively the nature of optimal strategies for a payoff kernel  $K(x,y)$  defined on the unit square satisfying  $K_{x_1 \dots x_n}(x,y) \geq 0$  with  $n$  partial derivatives taken with respect to  $y$ . We present a complete analysis for  $n \leq 4$ . However, the method employed easily extends and will enable one, by enumerating cases, to analyze the situation for general  $n$ . Specifically, it is shown that for  $n = 3$  and  $n = 4$  the maximizing player has optimal strategies involving respectively at most 3 points and at most 4 points of increase. For general  $n$ , it can be shown that the maximizing player has optimal strategies using at most  $n$  points. For the minimizing player the statement of the nature of an optimal strategy is more precise. There always exist for the general case optimal solutions using at most  $\frac{n}{2}$  points, with the understanding that the end points 0 or 1 when used are each counted only half. For example when  $n$  is odd, then  $\frac{n}{2}$  is a half integer and hence must use a single end point if a full optimal strategy exists employing  $\frac{n}{2}$  points. This counting procedure applies only to the minimizing strategies. ( )

## §1. Optimal strategies for player II.

We assume throughout this section that  $K_{y_1 \dots y_n}(x,y) \geq 0$  in the unit square with  $n$  partial derivatives taken with respect to  $y$  and that  $K$  possesses continuous  $n^{\text{th}}$  partial derivatives.

Lemma 1. If  $K_{y\dots y}(x,y) \geq 0$ , then there exists an optimal strategy for player II with at most  $\frac{n}{2}$  points of increase.

Remark 1. It is sufficient to prove Lemma 1 when  $K_{y\dots y}(x,y) \geq \delta > 0$ . Indeed, if we perturb  $K$  by  $K^n(x,y) = K(x,y) + \frac{1}{n} L(x,y)$  where  $L_{y\dots y}(x,y) \geq \delta > 0$ , then  $K^n(x,y)$  converges uniformly to  $K(x,y)$  and  $K^n_{y\dots y}(x,y) \geq \frac{1}{n} \delta \geq 0$  for each  $n$ . The function  $L(x,y)$  can be chosen to be an  $n^{\text{th}}$  degree polynomial in  $x$  and  $y$  with the coefficient of  $y^n$  a strictly positive polynomial for  $x$  ranging in the unit interval. Each kernel  $K^n$  possesses an optimal solution  $G_n$  for player II of at most  $\frac{n}{2}$  points in the spectrum. We can select a limit distribution  $G_0$  which also involves at most  $\frac{n}{2}$  points of increase. Similarly, let  $F$  denote any limit distribution of optimal strategies  $F_n$  of the games  $K^n(x,y)$ . It follows easily that  $G_0$  and  $F$  are optimal strategies for players II and I respectively for the game with kernel  $K(x,y)$ .

Proof of Lemma 1. In view of remark 1 we may assume that  $K_{y\dots y}(x,y) > 0$  for  $x,y$  traversing the unit square. Let  $F_0(x)$  denote an optimal strategy for player I, then the hypothesis and bounded convergence easily yields that  $h(y) = \int_0^1 K(x,y) dF_0(x)$  has  $h^{(n)}(y) > 0$ . This implies that  $h(y)$  can achieve a value at most  $n$  times since otherwise there exists a  $y_0$  with  $h^n(y_0) = 0$ . Consequently, the minimum  $m$  of  $h(y)$  can only be achieved at most  $\frac{n}{2}$  times, end points being counted half as described in the introduction; for otherwise  $m + \epsilon$  with  $\epsilon$  sufficiently small would be taken on at least  $n + 1$  times. Since an optimal strategy for the minimizing player can only concentrate at these points, the conclusion of the lemma is now evident.

## § 2. Optimal strategies for player I with $n = 1$ and $2$ .

The case  $n = 1$  yields a saddle point. In fact if  $x_0$  denotes a point where  $K(x_0, 0) = \max_x K(x, 0)$ , then it is easily verified that  $x = x_0$  and  $y = 0$  is a saddle point.

Lemma 2. Let  $X$  be compact and  $Y$  an  $n$  dimensional simplex. If  $M(x, y)$  is a continuous payoff kernel defined on  $X \times Y$  which for each  $x$  is a convex function in  $y$ , then there exists an optimal strategy for player I involving at most  $n + 1$  points of increase.

This lemma comprises the essential result of [1]. For a more direct and illuminating proof see a forthcoming paper by the author on the general theory of infinite games.

The content of lemma 2 contains the case of  $n = 2$  being analyzed in this paper. Player II has a pure optimal strategy while player I possesses an optimal solution using at most 2 points in its spectrum.

## § 3. Optimal strategies for player I with $n = 3$ .

Throughout this section we assume that  $K_{yyy}(x, y) \geq 0$  and we establish the following theorem.

Theorem 1. If  $K$  possesses continuous third partial derivatives and  $K_{yyy}(x, y) \geq 0$  for  $0 \leq x, y \leq 1$ , then player I has an optimal strategy  $F$  with at most 3 points of increase.

The proof of this theorem will consist in analyzing the various possibilities. It is no loss of generality to suppose that  $K_{yyy}(x, y) > 0$ . An argument analogous to that used in remark 1 establishes this fact. Moreover, any optimal minimizing strategy involves at most  $3/2$  points. The various possible minimizing strategies are therefore of the following forms, where  $I_y$  denotes a pure distribution with full weight at  $y$ : (a)  $\alpha I_0 + (1-\alpha) I_{y_0}$

with  $0 < y_0 < 1$ , (b)  $\alpha I_0 + (1-\alpha)I_1$ , (c)  $I_{y_0}$  with  $0 < y_0 < 1$ , (d)  $I_0$  and (e)  $I_1$ . The strategies listed above present an enumeration of all the possible ways of obtaining  $\frac{1}{2}$ , 1, or  $1\frac{1}{2}$  points, with the exception of one. It remains only to show why an optimal strategy of the form  $\alpha I_{y_0} + (1-\alpha)I_1$  with  $0 < y_0 < 1$  cannot occur.

(A) In fact, if  $F(x)$  is optimal for player I, then  $H(y) = \int_0^1 K(x,y)dF(x)$  has the properties that  $h(1) = h(y_0) = v$ ,  $h(y) \geq v$ , and  $h'''(y) > 0$ . This implies that  $h$  is convex in the neighborhood of  $y_0$  where  $h$  attains a minimum and further, since  $h(1) = v$ , that  $h$  is concave at  $y_1 > y_0$ . Thus  $h''(y_0) > 0$  and  $h''(y_1) < 0$ . But since  $h''(y) > 0$ , it follows that  $h''(y)$  is increasing which is incompatible with the preceding remark.

In general, once the yield  $h(y) = \int_0^1 K(x,y)dF(x)$  is convex then it cannot turn back and become concave.

To establish the assertion of Theorem 1, we shall show for each of the possible forms of an optimal solution for player II that player I possesses an optimal strategy of the desired type.

Case (a). Since  $G_0 = \alpha I_0 + (1-\alpha)I_{y_0}$  is optimal for player II, any value  $x$  occurring with positive probability in an optimal  $F$  strategy must imply that  $\int_0^1 K(x,y)dG_0(y) = v$ . Let  $X$  be the set of all  $x$  for which  $\int_0^1 K(x,y)dG_0(y) = v$ . Evidently, the set  $X$  is compact. We now introduce the following auxiliary game defined for  $x$  in  $X$  and  $z$  ranging over a 2 dimensional simplex  $z$  spanned by  $z_1, z_2$ , and  $z_3$ . For  $x$  in  $X$ , set  $M(x, z_1) = v_1 - v$ , where  $v_1$  is the value at 1 of the tangent line to  $K(x,y)$  at  $y_0$ ,  $M(x, z_2) = v_2 - v$ , where  $v_2$  is the value at 0 of the tangent line to  $K(x,y)$  at  $y_0$ , and  $M(x, z_3) = K(x, 0) - v$ . Extend  $M(x, z)$  linearly over  $z$  for each  $x$ . It is easy to verify that the hypothesis of lemma 2 are satisfied. Also if  $F$  is optimal for the game  $K(x,y)$ , then

$\int_X M(x, z) dF(x) = 0$ . Therefore, there exists an optimal  $F^*(x) = \lambda_1 I_{x_1} + \lambda_2 I_{x_2} + \lambda_3 I_{x_3}$  for which  $\int_X M(x, z) dF^*(x) \geq \text{value} \geq 0$ . We consider  $h(y) = \int_0^1 K(x, y) dF^*(x)$ . The interpretation of the game  $M(x, z)$  shows that the tangent to  $h(y)$  at  $y_0$  lies above  $v$  at 0 and 1. Thus  $h(y_0) \geq v$ . Also,  $h(0) \geq v$ . However, since  $x_1, x_2$ , and  $x_3$  belong to  $X$ , we obtain that  $v \leq \int_0^1 h(y) dG_0(y) = \int_0^1 dF^*(x) \int_0^1 K(x, y) dG_0(y) = v$ . This yields that  $h(0) = h(y_0) = v$  and hence the slope at  $h'(y_0) = 0$ . The argument of paragraph (A) shows that  $h(y)$  is convex at  $y_0$  and  $h(y) \geq v$  throughout.

Case (b). Let  $X$  consist of all  $x$  for which  $v = \int_0^1 K(x, y) dG_0(y)$  where  $G_0(y)$  is optimal and of the form  $\alpha I_0 + (1-\alpha)I_1$ . Evidently,  $X$  is compact. Let  $M(x, z)$  be defined over the set  $X$  and the two dimensional simplex  $z$  spanned by  $z_1, z_2$ , and  $z_3$  defined as follows. Put  $M(x, z_1) = K(x, 0) - v$ ,  $M(x, z_2) = K(x, 1) - v$  and  $M(x, z_3) = -K_y(x, 1)$  with  $M(x, z)$  extended linearly over  $z$ . If  $F_0(x)$  is any optimal strategy for player I of the game corresponding to the payoff kernel  $K(x, y)$ , then  $\int_X M(x, z) dF(x) \geq 0$ . In view of lemma 2 we can find an optimal strategy  $F^*(x)$  using at most 3 points  $x$  in  $X$ . If  $h(y) = \int_0^1 K(x, y) dF^*(x)$ , then  $h(0) \geq v$ ,  $h(1) \geq v$  and  $h'(1) \leq 0$ . Since  $G_0(y)$  concentrates only at 0 and 1 and  $\int_0^1 dG_0(y) \int_0^1 K(x, y) dF^*(x) = v$  we conclude that  $h(0) = h(1) = v$ . Furthermore  $h'(y)$  is negative at 1 and  $h'''(y) > 0$  requires therefore that  $h(y)$  is greater than  $v$  throughout the unit interval.

Case (c). In this case an optimal minimizing strategy exists concentrating purely at point  $y_0$ . Let  $X$  comprise all  $x$  where  $K(x, y_0) = v$ . Again, we introduce the auxiliary game  $M(x, z)$  with  $x$  in  $X$  and  $z$  in the two dimensional simplex. Put  $M(x, z_1) = +K_y(x, y_0)$ ,  $M(x, z_2) = -K_y(x, y_0)$ , and

$M(x, z_3) = K(x, 0) - v$  with the usual linear extension. An analogous reasoning produces an optimal strategy  $F^* (= \alpha I_{x_1} + \beta I_{x_2} + \gamma I_{x_3})$  with  $\int_X M(x, z) dF^*(x) \geq 0$ . The function  $h(y) = \int_0^1 K(x, y) dF^*(x)$  has the properties that  $h(y_0) = v$ ,  $h'(y_0) = 0$  and  $h(0) \geq v$ . The argument in (A) and  $h(0) \geq v$  show that  $h$  is convex at  $y_0$  and since  $h(0) \geq v$  we easily find that  $h(y) \geq v$  for  $0 \leq y \leq 1$ .

Case (d). The form of  $G_0$  is  $I_0$ . We select for  $X$  all  $x$  where  $K(x, 0) = v$ . We subdivide the analysis of (d) into three subcases. (d<sub>1</sub>) Let there exist an optimal strategy for player I with  $m(y) = \int_0^1 K(x, y) dF_0(x)$  such that  $m(0) = v$ ,  $m(y) \geq v$ ,  $m'(y_0) = 0$  and  $m''(y_0) > 0$  for some  $y_0$  with  $0 \leq y_0 < 1$ . We define  $M(x, z)$  for  $x$  in  $X$  and  $z$  in  $Z$  (two-dimensional simplex) as follows:  $M(x, z_1) = v_1 - v$ , where  $v_1$  is the value at 1 of the tangent line to the curve  $K(x, y)$  at  $y_0$ ,  $M(x, z_2) = v_2 - v$ , where  $v_2$  is the value at 0 of the tangent line to the curve  $K(x, y)$  at  $y_0$ , and  $M(x, z_3) = K_{yy}(x, y_0)$  and  $M(x, z)$  linearly extended over  $z$ . It is immediately verified that  $\int_X M(x, z) dF_0(x) \geq 0$ . Lemma 2 thus guarantees an  $F^*(x) = \alpha I_{x_1} + \beta I_{x_2} + \gamma I_{x_3}$  with  $\int_X M(x, z) dF^*(x) \geq 0$ . Let  $h(y) = \int_0^1 K(x, y) dF^*(x)$ . It follows that  $h(0) = v$ ,  $h(y)$  is convex at  $y_0$  and the tangent line at  $y_0$  to  $h(y)$  lies completely above the horizontal line at height  $v$ . Consequently, the whole convex portion of  $h(y)$  lies above  $v$  and since  $h(0) = v$ , we get throughout  $h(y) \geq v$ . This is true since once  $h(y)$  is convex it cannot become concave afterwards, (see the argument of (A)). (d<sub>2</sub>) Suppose there exists an optimal strategy  $F_0(x)$  such that  $m(y) = \int_0^1 K(x, y) dF_0(x)$  is concave for  $0 \leq y \leq 1$  and  $m(y) \geq v$ . The auxiliary game we set up becomes  $M(x, z_1) = -K_{yy}(x, 1)$  and  $M(x, z_2) = K(x, 1) - v$ . In this case  $Z$  is one dimensional and we can find



an optimal strategy  $F^*$  using two values of  $x$  in  $I$  (the values  $x$  where  $K(x,0) = v$ ). Also,  $h(y) = \int_0^1 K(x,y) dF^*(x)$  satisfies  $h''(1) \leq 0$ ,  $h(0) = v$  and  $h(1) \geq v$ . It follows easily from  $h'''(y) > 0$  that  $h$  remains concave throughout the interval and hence  $h(y) \geq v$ . (d<sub>3</sub>) If  $m(y)$  is convex at 0, then in a similar manner it is easy to find an optimal strategy of the desired type.

Case (e). This can be analyzed similarly to case (d). Since these five cases exhaust all the possibilities the proof of Theorem 1 is now complete.

#### §4. Optimal strategies for player I with $n = 4$ .

Throughout this section we assume that  $K_{yyyy}(x,y) \geq 0$ . The following result is demonstrated.

Theorem 2. If  $K$  possesses continuous fourth partial derivatives and  $K_{yyyy}(x,y) \geq 0$  for  $0 \leq x, y \leq 1$ , then player I has an optimal strategy with at most 4 points of increase.

The method of proof of this theorem is analogous to that employed in Theorem 1. Again, without restricting generality we may assume that  $K_{yyyy}(x,y) > 0$ . Furthermore an optimal strategy for player II involves at most  $\frac{4}{2} = 2$  points. The various possible minimizing strategies are therefore of the form (a)  $\alpha I_{y_0} + (1-\alpha) I_{y_1}$ , with  $0 < y_0 < y_1 < 1$ , (b)  $\alpha I_1 + (1-\alpha) I_{y_0}$  with  $0 < y_0 < 1$ , (c)  $\alpha I_1 + (1-\alpha) I_{y_0}$  with  $0 < y_0 < 1$ , (d)  $\alpha I_0 + (1-\alpha) I_1$  (e)  $I_{y_0}$  with  $0 < y_0 < 1$ , (f)  $I_0$ , and (g)  $I_1$ . We have omitted one additional possible strategy with total weight 2; namely  $\alpha I_0 + \beta I_{y_0} + (1-\alpha-\beta) I_1$ , where  $0 < y_0 < 1$ . This form for an optimal strategy is not possible.

(B) Indeed, let us suppose that player II has an optimal procedure of the above form. Let  $F_0(x)$  be optimal for player I, then  $m(y) = \int_0^1 K(x,y) dF_0(x)$

has the following properties:  $M(0) = v$ ,  $m(1) = v$ ,  $m(y_0) = v$ ,  $m'(y_0) = 0$  and  $m(y) \geq v$  for  $0 \leq y \leq 1$ . It follows then that  $m$  is convex at  $y_0$  while concave at 0 and at 1. Thus  $m''(0) < 0$ ,  $m''(1) < 0$ , and  $m''(y_0) > 0$ . This contradicts the fact that  $m''(y)$  is convex which is required as  $m'''(y) > 0$ .

We shall frequently use this general fact that a yield  $m(y)$  of any strategy  $F$  i.e.  $m(y) = \int_0^1 K(x,y)dF(x)$  cannot be concave in two portions and convex in between.

The proof of Theorem 2 will consist in verifying for each of the possible forms of the optimal strategies for player II that an optimal strategy for player I of the desired type exists.

Case (a). Let the minimizing player have an optimal solution of the form  $G_0 = \alpha I_{y_0} + (1-\alpha)I_{y_1}$ , where  $0 < y_0 < y_1 < 1$ . Let the set  $X$  consist of all  $x$  for which  $\int_0^1 K(x,y)dG_0(y) = v$ . We construct now the following auxiliary game  $M(x,z)$  defined for  $x$  in  $X$  and  $z$  in  $Z$  where  $Z$  is a 3 dimensional simplex spanned by the points  $z_1, z_2, z_3$  and  $z_4$ . For  $x$  in  $X$ , set  $M(x,z_1) = v_1 - v$ , where  $v_1$  is the tangent line to the curve  $K(x,y)$  at  $y_0$  evaluated at 0,  $M(x,z_2) = v_2 - v$ , where  $v_2$  is the tangent line to the curve  $K(x,y)$  at  $y_0$  evaluated at 1, and  $M(x,z_3)$  and  $M(x,z_4)$  are defined similarly in terms of curve  $K(x,y)$  at the point  $y_1$ . The function  $M(x,z)$  is defined over  $Z$  by linear extension. The kernel  $M(x,z)$  satisfies all the conditions of lemma 2. Furthermore, if  $F_0(x)$  is optimal for player I and the game  $K(x,y)$ , then  $\int_X M(x,z)dF_0(x) = 0$ . Lemma 2 provides an optimal strategy  $F^*(x)$  with at most 4 points in its spectrum such that  $\int_X M(x,z)dF^*(x) \geq 0$ . Put  $h(y) = \int_0^1 K(x,y)dF^*(x)$ . The optimal nature of  $F^*(x)$  for the game  $M(x,z)$  implies that the tangent lines

to  $h(y)$  at  $y_0$  and  $y_1$  lie everywhere above the height  $v$ . Consequently,  $h(y_0) \geq v$  and  $h(y_1) \geq v$ . Hence,  $\int_0^1 h(y) dG_0(y) \geq v$ . But since every  $x$  involved in  $F^*(x)$  is included in  $X$ , we conclude that  $\int_0^1 dF^*(x) \int_0^1 K(x,y) dG_0(y) = v$ . Thus,  $h(y_0) = h(y_1) = v$ . Therefore also,  $h'(y_0) = h'(y_1) = 0$ . In view of the remark (B), we deduce that  $h(y) \geq v$  throughout the unit interval.

Case (b). For this case we construct the auxiliary game over  $X$  and  $Z$  as follows: The values of  $M(x, z_1)$  and  $M(x, z_2)$  are exactly as in Case (a). Put  $M(x, z_3) = K(x, 0) - v$  and  $M(x, z_4) = +K_y(x, 0)$  and  $M(x, z)$  for  $z$  in  $Z$  is defined by linear extension. The set  $X$  consists of all  $x$  for which  $\int K(x, y) dG_0(y) = v$  where  $G_0(y) = \alpha I_0 + (1-\alpha) I_{y_0}$ . By lemma 2 the game  $M(x, z)$  with  $x$  in  $X$  and  $z$  in  $Z$  has an optimal strategy  $F^*(x)$  with at most 4 points of increase such that  $\int_X M(x, z) dF^*(x) \geq 0$ . This gives for  $h(y) = \int_0^1 K(x, y) dF^*(x)$  that  $h(0) \geq v$  and  $h(y_0) \geq v$ . It can be deduced in a similar manner to case (a) that  $h(0) = h(y_0) = v$  and  $h'(y_0) = 0$  while  $h'(0) \geq 0$ . With the aid of remark (B) we conclude easily that  $h(y) \geq v$  for  $0 \leq y \leq 1$ .

Case (c). If the minimizing optimal strategy has the form  $G_0 = \alpha I_1 + (1-\alpha) I_{y_0}$ , then an argument similar to that employed in case (b) applies here to furnish the desired kind of optimal strategy for player I.

Case (d). Let  $G_0(y) = \alpha I_0 + (1-\alpha) I_1$  and let  $X$  consist of all  $x$  in the unit interval for which  $\int K(x, y) dG_0(x) = v$ . Put for  $x$  in  $X$   $M(x, z_1) = K(x, 0) - v$ ,  $M(x, z_2) = K_y(x, 0)$ ,  $M(x, z_3) = K(x, 1) - v$  and  $M(x, z_4) = -K_y(x, 1)$  with a linear extension. Again, one can find an optimal strategy  $F^*(x)$  involving at most 4 points for which  $\int M(x, z) dF^*(x) \geq 0$ . It follows easily for  $h(y) = \int_0^1 K(x, y) dF^*(x)$  that  $h(0) \geq v$ ,  $h(1) \geq v$ ,

$h'(0) \geq 0$  and  $h'(1) \leq 0$ . Employing the definition of  $X$  yields that  $h(0) = h(1) = v$ . On account of remark (B) it follows that  $h(y) \geq v$  for  $y$  in the unit interval.

Case (e). The form of the minimizing solution is  $I_{y_0}$  with  $0 < y_0 < 1$ . We subdivide this case into four subcases. (e<sub>1</sub>). Suppose there exists an optimal  $F_0(x)$  for which  $m(y) = \int_0^1 K(x,y) dF_0(x)$  is convex for all  $y$  in the unit interval. Let  $X$  be composed of all  $x$  where  $K(x,y_0) = v$ . We now define a new game  $M(x,z)$  over  $X \times Z$  as follows. For  $x$  in  $X$ , set  $M(x,z_1) = v_1 - v$ , where  $v_1$  is the tangent line to the curve  $K(x,y)$  at  $y_0$  evaluated at 0, and  $M(x,z_2) = v_2 - v$ , where  $v_2$  is the tangent line to the curve  $K(x,y)$  at  $y_0$  evaluated at 1. Finally for the segment  $[z_3, z_4]$  which we choose to be the unit interval  $[0,1]$ , we define  $M(x,z) = K_{yy}(x,z)$ . Since the fourth derivative  $K_{yyyy} \geq 0$ , we get that  $M(x,z)$  is convex on the segment  $[z_3, z_4]$  for each  $x$ . The function  $M(x,z)$  is extended linearly over the rest of the simplex  $Z$ . It obviously satisfies the requirements of lemma 2. Furthermore, the nature of  $m(y)$  implies that  $\int_X M(x,z) dF_0(x) \geq 0$ . On account of lemma 2, we can select an optimal strategy  $F^*(x)$  which involves at most 4 points in its spectrum for which  $\int_X M(x,z) dF^*(x) \geq 0$ . Let  $h(y) = \int_0^1 K(x,y) dF^*(x)$ . An interpretation of the auxiliary game introduced yields that  $h''(y) \geq 0$ ,  $h'(y_0) = 0$  and  $h(y_0) = v$ . Consequently,  $h(y) \geq v$  for  $0 \leq y \leq 1$ . (e<sub>2</sub>). Suppose there exists an optimal strategy  $F_0(y)$  for which  $m(y) = \int_0^1 K(x,y) dF_0(x)$  is concave at 0. The set  $X$  is chosen as we did before for (e<sub>1</sub>). We construct  $M(x,z_1)M(x,z_2)$  as before while  $M(x,z_3) = K(x,0) - v$  and  $M(x,z_4) = -K_{yy}(x,0)$  with  $M(x,z)$  extended linearly over  $Z$  for each  $x$  in  $X$ . An optimal strategy using at most 4 points for the game corresponding to  $M(x,z)$  turns out to be an optimal procedure also for  $K(x,y)$ . The details are similar to the preceding cases. A

symmetrical argument takes care of the possibility when  $m(y)$  is concave at 1. ( $e_3$ ). We assume now that there exists an optimal strategy  $F_0(y)$  for which  $m(y) = \int_0^1 K(x,y) dF_0(x)$  has the following properties:  $m'(y_0) = 0$ ,  $m(y_0) = v$ ,  $m(y) \geq v$  for  $0 \leq y \leq 1$ , there exists a  $0 < y_1 < y_0$  for which  $m''(y_1) > 0$  and  $m'(y_1) = 0$ . Let  $X$  comprise those  $x$  for which  $K(x, y_0) = v$ . Let  $Z$  be a 4 dimensional simplex spanned by  $z_1, z_2, z_3, z_4$  and  $z_5$ . Let for  $x$  in  $X$   $M(x, z_1)$  and  $M(x, z_2)$  be given as before. Also, put  $M(x, z_3) = v_3 - v$ , where  $v_3$  is the tangent line to the curve  $K(x, y)$  at  $y_1$  evaluated at 0,  $M(x, z_4) = v_4 - v$ , where  $v_4$  is the tangent line to the curve  $K(x, y)$  at  $y_1$  evaluated at 1, and  $M(x, z_5) = K_{yy}(x, y_1)$ . Furthermore  $M(x, z)$  is extended linearly over  $Z$ . It follows easily that  $\int_X M(x, z) dF_0(x) \geq 0$ . We now verify that the optimal strategies for player II for the game with payoff  $M(x, z)$  cannot possess an interior pure strategy  $I_{z_0}$ . Otherwise,  $\int_X M(x, z) dF_0(x) = 0$  for it is simple to show that the value is zero and therefore  $m(y) = \int K(x, y) dF_0(x)$  satisfies  $m(y) \geq v$  throughout and  $m(y_1) = v$ ,  $m'(y_1) = 0$  and  $m''(y_1) = 0$  which contradicts the assumption made on  $m(y)$ . In view of [1] and the corollary to theorem 4 of [2], we can conclude, since  $I_{z_0}$  is on the boundary of  $Z$ , that there exists an optimal strategy  $F^*(x)$  using at most 4 points. Let  $h(y) = \int_0^1 K(x, y) dF^*(x)$ . It follows that  $h'(y_0) = 0$ ,  $h(y_0) = v$ , the tangent line to  $h(y)$  at  $y_1$  lies above  $v$  for  $0 \leq y \leq 1$  and  $h(y)$  is convex at  $y_1$ . Now with the aid of remark (B) it is easy to show that  $h(y) \geq v$  throughout the unit interval. A symmetrical argument works if the loops of  $m(y)$  take place on the side toward 1.

It remains only to consider the case where  $m(y) = \int K(x, y) dF_0(x)$  satisfies the same properties as in ( $e_3$ ) except that  $y_1 = 0$ . We then define

$M(x, z_1)$  and  $M(x, z_2)$  as before while  $M(x, z_3) = K(x, 0) - v$  and  $M(x, z_4) = K_y(x, 0)$  with  $x$  any point in the set  $X$  where  $K(x, y_0) = v$ . A simple analysis with the help of the game  $M(x, z)$ , shows in this circumstance the existence of an optimal strategy of the desired kind.

(d<sub>4</sub>) The final case consists of a solution  $F_0(x)$  such that  $m(y) = \int_0^1 K(x, y) dF_0(x)$  is convex at  $y_0$  and at 0 in such a way that  $m'(0) \geq 0$ . The auxiliary game used here is the same employed in case (b). An optimal strategy  $F^*(x)$  using at most 4 points exists with  $h(y) = \int K(x, y) dF^*(x)$  such that  $h'(y_0) = 0$ ,  $h(y_0) = v$ ,  $h(0) \geq 0$  and  $h'(y_0) \geq 0$ . It follows on account of (A) that  $h(y) \geq v$  in the unit interval. A symmetrical analysis applies if  $m(y)$  is convex at 1 with  $m'(y) \leq 0$ .

Case (f) and Case (g). The arguments for these two cases are similar to the preceding.

The proof of theorem 2 is now complete in view of the fact that we have treated every possibility. It is interesting to note that the proof of Theorem 2 introduced some new techniques in order to exhibit the desired strategies. In particular, we emphasize the proof of case (e).

In a future paper we intend to present a generalization of this result to the case where  $x$  ranges over a compact space and  $y$  traverses a  $k$  dimensional set with the condition  $K_{y \dots y}(x, y) \geq 0$  replaced by the requirement that the  $n^{\text{th}}$  term of the Taylor expansion in  $y$  should be non-negative. Also, the question of uniqueness of the set of optimal strategies can be analyzed.

## § 5. Dimensional relations.

In this section we analyze further properties of the optimal strategies for the two cases considered in sections 3 and 4. We deal first with the case where  $K_{yy}(x,y) > 0$ . Here any optimal minimizing strategy must be confined at most to two points, 0 and  $y_0$  or 0 and 1, and we can therefore speak about the dimension of the set of solutions for the minimizing players. The two possible cases are 0 and 1 dimensional sets. If the spectrum for the optimal minimizing strategies is restricted to one point, then clearly the solution is unique for player II.

Case (a). Let us suppose that player II has a one dimensional set of optimal strategies mixing the pure strategies 0 and  $y_0$ , with  $0 < y_0 < 1$ . There exist at least two optimal strategies of the form  $G = \lambda I_0 + (1-\lambda)I_{y_0}$  and  $G' = \lambda' I_0 + (1-\lambda')I_{y_0}$  with  $\lambda' \neq \lambda$ . We consider the set  $X$  of all  $x$  for which  $\int K(x,y)dG(y) = \int K(x,y)dG'(y) = v$ . Explicitly, we obtain

$$\lambda [K(x,0) - v] + (1-\lambda) [K(x,y_0) - v] = 0$$

$$\lambda' [K(x,0) - v] + (1-\lambda') [K(x,y_0) - v] = 0.$$

As  $\lambda \neq \lambda'$ , we find that  $K(x,0) = K(x,y_0) = v$ . We now construct the auxiliary game where  $z$  consists of a one dimensional simplex spanned by  $z_1$  and  $z_2$ . Put for  $x$  in  $X$   $M(x,z_1) = -K_y(x,y_0)$ ,  $M(x,z_2) = K_y(x,y_0)$  and let  $M(x,z)$  be defined on the remaining points of  $z$  by linear extension. There exists an optimal  $F^*$  using only two points at most. In view of the nature of  $X$  it is easily verified that  $F^*$  is an optimal strategy for the game given by  $K(x,y)$ .

Case (b). We assume that player II has a one dimensional set of strategies using only the points 0 and 1. It can be shown as above that there exists

an optimal maximizing strategy consisting of at most two points of increase.

We have thus demonstrated the following theorem:

Theorem 3. If  $K$  satisfies  $K_{yyy}(x,y) > 0$  and player II possesses a one dimensional set of optimal strategies then player I can find a solution using at most two points of increase.

In a similar manner we can obtain

Theorem 4. If  $K$  satisfies  $K_{yyyy}(x,y) > 0$  and player II possesses a one dimensional set of optimal strategies, then player I can find an optimal strategy with at most three points in the spectrum.

Every case is easily handled but one. Suppose the minimizing strategy has a one dimensional set of solutions using  $y_0$  and  $y_1$  with  $0 < y_0 < y_1 < 1$ . Let  $X$  consist of all  $x$  where  $K(x,y_0) = K(x,y_1) = v$ . An argument as in Theorem 3 shows that these are the only points  $x$  which need be considered. We construct the following auxiliary game defined over  $X$  and  $z$  where  $z$  is a two dimensional simplex. Put  $M(x,z_1) =$  the tangent line to  $K(x,y)$  at  $y_0$  evaluated at 1 minus the tangent line to  $K(x,y)$  at  $y_1$  evaluated at 1.  $M(x,z_2) =$  the same as above except the evaluation takes place at 0 and  $M(x,z_3) =$  the tangent line to  $K(x,y)$  at  $y_1$  evaluated at  $\frac{y_0+y_1}{2}$  minus  $v$ . If  $F$  is optimal for the game corresponding to  $K(x,y)$  then  $\int_X M(x,y) dF(x) = 0$ . Thus lemma 2 provides an optimal strategy  $F^*$  using at most 3 points of  $X$  with  $\int M(x,y) dF^* \geq 0$ . One can easily verify that  $F^*$  is optimal for  $K(x,y)$ . This completes the proof.

In the general case where  $K_{y \dots y}(x,y) > 0$  for  $n$  partial derivatives with respect to  $y$ , it can be shown that if the  $y$  player has a  $k$  dimensional set of solutions, then the  $x$  player can find an optimal strategy involving at most  $n-k$  points in its spectrum. We omit the proof since it is only long and tedious.



## § 6. An application.

We close this note with an application of Theorems 1 and 2 to a class of matrices. First, it is important to note that the conclusions of Theorems 1 and 2 remain valid when  $x$  traverses any compact set and where  $y$  ranges over the unit interval. We consider now a class of matrices  $(a_{ij})$   $1 \leq i \leq n$ ,  $1 \leq j \leq m$ , satisfying the requirement that  $\Delta_j^k a_{ij} \geq 0$  for each  $i$  where  $k$  is a fixed inte integer. It is easy to show that if there exists a finite sequence of numbers  $a_r$  ( $r = 0, 1, \dots, m$ ) which satisfy  $\Delta^k a_r \geq 0$  for each  $r$ , then the function  $f(y)$  defined as

$$f(r/m) = a_r, \quad r = 0, 1, \dots, m,$$

and defined by linear interpolation between the successive values of  $r/m$ , possesses the property that  $\Delta^k f(y) \geq 0$  with the difference increment taken to be  $1/m$ . We perform this extension for every row of the matrix and we secure a function  $K(x, y)$  with  $x$  ranging over a finite number of points  $i = 1, \dots, n$  and  $y$  over the unit interval. The conclusions of Theorem 1 and 2 remain valid for such a setup where  $k$  differentiations with respect to  $y$  are replaced by a process of  $k$  differences. Consequently, we can find optimal strategies of the game  $K(x, y)$  for player I and II using at most  $k$  points and  $k/2$  points respectively. Due to the linear nature of  $K(x, y)$  any point  $y_0$  can be obtained as a convex combination of two values  $r/m < y_0 < r+1/m$  with  $K(x, y_0) = \lambda K(x, r/m) + (1-\lambda)K(x, r+1/m)$ . Thus in terms of the original matrix both players I and II possess solutions employing  $k$  at most  $k$  rows and  $k$  columns.

Theorem 5. If  $a_{ij}$  is a matrix such that for each  $i$   $\Delta_j^k a_{ij} \geq 0$  for all  $j$ , then both players have solutions using at most  $k$  rows and columns respectively.

Finally, we remark that all the analogous results hold for the situations where  $K_{y\dots y}(x,y) \leq 0$ ,  $K_{x\dots x}(x,y) \geq 0$  and  $K_{x\dots x}(x,y) \geq 0$ .

#### REFERENCES

1. F. Bohnenblust, S. Karlin, and L. S. Shapley, "Games with Continuous Convex Payoff", *Annals of Math Studies*, No. 24, Dec. 1950.
2. S. Karlin and L. S. Shapley, "Applications of a Theorem on Convex Functions", *Annals of Math.* 1950.